SOME MEASURE THEORY ON STACKS OF NETWORKS

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Abstract. By counting symmetries of graphs carefully (or equivalently, by regarding moduli spaces of graphs as zero-dimensional orbifolds), certain measures on these collections (elsewhere called 'exponential random graphs') can be reinterpreted, with the aid of special cases of Wick's theorem, as Feynman-style measures on the real line. Analytic properties of the latter measures can then be studied in terms of phase transitions – in particular, in models for spaces of scale-free trees.

Introduction: This paper is an attempt at an essentially elementary account of a single example, which provides evidence for the emergence of an interesting statistical mechanics of networks.

For more than fifty years physicists have elaborated techniques introduced by Feynman, which organize calculations of probabilities on spaces of particle histories as sums over graphs. More recently [2], researchers have begun to use these methods in reverse, to attack problems in enumerative combinatorics by analytical means.

Here [§2.3] such (relatively familiar) ideas are formulated as an analog of the Fourier transform, as a way of evaluating sums over certain spaces of graphs (known in the literature [15] as exponential random measures) as integrals over the real line, with respect to certain Feynman/Gibbs-style measures. Collections of graphs or networks seem intuitively very discrete, but the transformed sums have interesting (and accessible) analytic properties, which can sometimes be described in ways similar to the phase transitions studied in thermodynamics.

The first section below reviews standard background material, but it contains a significant technical point: sums over graphs are really integrals over zero-dimensional moduli spaces, and if we treat orbifold points respectfully (ie by weighting their symmetries appropriately, by

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what physicists call Faddeev - Popov determinants) we get better formulas. Similar issues appear in other apparently discrete contexts; for example, in Siegel's mass formula for quadratic lattices [18 VII §6.5 (remark)].

Without compelling examples this would be empty formalism. Section three sketches the basic properties of a model (developed by physicists [3,4,12] interested in (among other things) the statistics of long-chain polymers) for a phase change in an ensemble of scale-free trees (ie having nodes of valence n occurring with power-law probability). This model has some remarkable features: its phase transition is of high (ie fourth) order, and its critical exponents depend to some extent on the parameters of the model. I hope this note will help make it, and its many generalizations, accessible for the closer study it deserves.

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1. The stack of weighted graphs

This section assembles standard facts and notation from graph theory:

1.1.0 The very familiarity of graphs can lead to confusion. I will be concerned with **finite, abstract** graphs; such a thing can be defined [8] to be a finite set G with an involution σ, together with a retraction $t : G \to G^\sigma$ onto the fixed point set of the involution. Elements of G^σ are the *vertices* of the graph, while the elements of the complement $G - G^\sigma$ are said to be its *half-edges*; the quotient of this set by the involution is the set of edges. The *valence* of a vertex $v \in G^\sigma$ is the cardinality $k(v)$ of $t^{-1}(v)$, ie the number of its incident edges; I will always assume this is positive. A vertex of valence one defines a *terminal edge*, or *leg*, of the graph; the remaining elements of G^σ comprise its set $\mathrm{Ver}^0(G)$ of *internal nodes*. (Legs will always be terminal.)

These definitions are somewhat counter-intuitive but they have some virtue; a good test case is the problem of enumerating the possible labellings of a graph with one edge and two vertices.

Graphs define a category, the morphisms being maps of sets with involution, compatible with the specified retractions; but isomorphisms will play a particularly important role in this note. Abstract graphs have natural geometrical realizations as one-dimensional simplicial complexes; we will not use them here, but they are important in generalizations in which the graphs map to interesting configuration spaces [5].

1.1.1 I will always assume that an abstract graph G carries a specified order, or *labeling*, on its set of legs. I will also consider graphs *weighted* by a function w_G from the internal nodes of G to $\{0, 1, 2, \ldots\}$. I will write $\mathrm{Aut}(G)$ for the group of isomorphisms of G, compatible with its labeling and weight function (which will in general be suppressed from the notation). $\Gamma_\bullet$ (resp. $W\Gamma_\bullet$) will denote the stack (or groupoid, or zero-dimensional orbifold) of abstract (resp. weighted) graphs, subject to a stability condition on internal nodes (made precise below), with isomorphisms as maps. $|\Gamma_\bullet|$ and $|W\Gamma_\bullet|$ will be the associated sets of isomorphism classes of objects.

1.1.2 It is often easiest for technical purposes to work with connected graphs, and I will write Γ [resp $W\Gamma$] for the subcategories of such things. The genus, or first Betti number, of a connected graph is $g(G) = E - V + 1$, with E its total number of edges, and V the number of vertices; this is additive under disjoint union, but for some purposes the Euler characteristic $\chi(G)$ $(= 1 - g(G)$ for connected graphs) which is also additive, is more useful.

The number $\nu(G)$ of (external) legs, and the number $\varepsilon = E - \nu$ of internal edges, are other useful additive functions. The Euler characteristic (or genus), together with the number of legs, define a kind of bigrading on the category of graphs. More generally, the generalization

$$g_w(G) := \sum_{v \in \mathrm{Ver}^0(G)} w_G(v) + g(G)$$

of the genus is additive on weighted graphs, and defines a similar bigrading on $W\Gamma_\bullet$.

A connected G is **stable** with respect to its weight function, if $2(w_G(v) - 1) + k(v) > 0$ for each internal node [10 §2]; note that for unweighted graphs (ie with $w_G = 0$) this precludes nodes of valence two. I'll write

$|\Gamma(\chi,\nu)|$ for the set of isomorphism classes of connected graphs of Euler characteristic χ with ν legs, and so forth.

There are many other interesting naturally defined functions on these sets, such as the degree sequence

$$D(G) = \{\nu = d_1, d_2, \ldots, d_m\}$$

which assigns to G, the partition of $2E$ with d_k equal to the number of vertices of valence k. The distribution of weights, or numbers of legs, define similar functions on $|W\Gamma(\chi,\nu)|$.

The function which sends G, weighted or not, to its group $\mathrm{Aut}(G)$ of automorphisms is another important example, as is the related group of automorphisms which are allowed to change the labelings on the legs. When G is connected, this is just the product of $\mathrm{Aut}(G)$ with the symmetric group of permutations of the labels, but if not, things are more complicated; see below. Note that (labeled) trees have **no** automorphisms.

1.2 Graphs in general can be described in terms of the symmetric product of collections of connected graphs, but we will need to keep careful track of symmetries.

1.2.0 An ordered n-tuple of elements from a set X is an element of the n-fold Cartesian product X^n of X with itself. Permuting the order of these elements defines an action

$$\Sigma^n \times X^n \to X^n$$

of the symmetric group on this set, and the quotient X^n/Σ_n is the set of **un**ordered n-tuples of elements from X. The coproduct, or disjoint union,

$$\coprod_{n\geq 0} X^n/\Sigma_n \;:=\; \mathrm{SP}^\infty(X)$$

of these quotients is the free abelian monoid generated by X: its elements can be written as formal finite sums

$$\sum n_i\{x_i\}$$

in which the n_i are non-negative integers, and the x_i are (not necessarily distinct) elements of X; of course these formal sums are subject to various rules, such as $n\{x\}+\{x\} = (n+1)\{x\}$, etc. $\mathrm{SP}^\infty(X)$ is naturally graded by the degree $\sum n_i\{x_i\} \mapsto \sum n_i$.

1.2.1 The isomorphism class of a general element G of $\Gamma_\bullet$ can thus be written as a formal sum $\sum n_i[G_i]$, defined by the disjoint union of n_i

copies of (the isomorphism classes of) connected graphs G_i. In other words,

$$|\Gamma_\bullet| \;\cong\; \mathrm{SP}^\infty|\Gamma|$$

as graded sets, with the understanding that we now admit a 'vacuum' graph with no vertices, cf eg [4 Ch 7]. If, in the presentation of G as a sum, the indexed components are distinct, then its automorphism group

$$\mathrm{Aut}(G) \;\cong\; \prod(\Sigma_{n_i} \wr \mathrm{Aut}(G_i)) \;.$$

is a kind of wreath product. If $\nu = \sum n_i \nu_i = \sum k r_k$, where r_k is the number of components of G with precisely k terminal legs, then the group of automorphisms of G which are allowed to permute those legs will be a semidirect product of the restricted automorphism group with $\prod \Sigma_k^{r_k}$. I'll write

$$m(G,\nu) = \frac{\nu!}{\prod k!^{r_k}}$$

for the number of ways of labeling the legs of G.

1.2.2 Finally, it may be worth noting (since the symmetric product construction is not much used in analysis) that a measure on X pushes forward to define a measure on $\mathrm{SP}^\infty(X)$. A measure on connected (weighted) graphs thus extends naturally to define a measure on general (weighted) graphs.

The 'measures' of most interest in this paper will, however, take values in some field $\mathbb{R}((\kappa))$ of formal series; κ will play the role of a parameter like Planck's constant in some asymptotic expansion. In fact the formulas will usually include even more parameters.

2. A kind of Feynman transform

This note is concerned with measures on spaces of (weighted) graphs, which is a subject with a large and somewhat disconnected literature; in [15 §5, 16] the term 'exponential random graphs' is used for a class of examples somewhat wider than those which will be in focus here.

2.1 That general class of models employs a family ϵ_k of functions on (isomorphism classes of) graphs, together with corresponding parameters β_k, called 'inverse temperatures'; a graph G is then assigned probability

$$P_\beta(G) = \exp(-\sum \beta_k \epsilon_k(G))$$

(perhaps suitably normalized). If the ϵ's are additive (under disjoint union of graphs), these probabilities will be multiplicative, and their values on general graphs will be extended from their values on connected graphs as described in the preceding section.

That will be the case here, with some modifications: $\mathcal{P}_\beta$ will be defined by functions $\epsilon_{w,k}(G)$ which count the total number of vertices in G with weight w and valence $k > 1$, and I will also introduce parameters t, λ and κ to keep track of the number of legs, edges and Euler characteristic (which are all of course additive). The literature of exponential random graphs allows more complicated ϵ's, which count more general subconfigurations (various kinds of polygons, etc) of G; such data can be incorporated into measures of the form $F(G)\mathcal{P}_\beta(G)$.

In this paper the key difference involves the enumeration of graphs with symmetries. If we define

$$\mathcal{P}_\beta(G) \ = \ \frac{m(G,\nu)}{|\mathrm{Aut}(G)|} \exp(-\sum \beta_{w,k}\epsilon_{w,k}(G))$$

then our 'measure' $\mu_{\beta,\lambda}$ (actually taking values in the formal power series ring $\mathbb{R}((\kappa))[[t,\lambda]]$) will be defined by summing the function

$$G \mapsto \mathcal{P}_\beta(G)\kappa^{-\chi(G)}\lambda^{\varepsilon(G)}\frac{t^{\nu(G)}}{\nu(G)!}$$

over subsets of $|W\Gamma_\bullet|$. The extra combinatorial factors in the definition of $\mathcal{P}_\beta$ divide by the order of the group of extended automorphisms of the graph (allowed to permute the labels on the legs). Note that the order $|\mathrm{Aut}(G)|$ is not quite multiplicative under disjoint union.

2.2 The interest of this class of measures is that under certain circumstances they can be calculated, or at least described, in terms of formal measures defined by **local** interactions in a one-dimensional Euclidean space.

Feynman introduced certain expressions of the general form

$$\int f(x)\ \exp(-\kappa^{-1}\mathcal{L}(x))\ dx\ ,$$

interpreted as perturbed Gaussian integrals over infinite-dimensional spaces, and he showed that these could be in some sense evaluated in terms of formal series involving more and more complicated but finite-dimensional integrals; but it was the physicist Gian-Carlo Wick who systematized the combinatorics behind those calculations. This section is concerned with a drastically simplified case of Wick's result,

which provides a rigorous asymptotic expansion for expressions like the integral above. In this account the domain of integration will be taken to be one-dimensional [9, 10], but the techniques below generalize quite naturally to 'non-linear sigma models', in which the graphs are mapped by the parameter x to some kind of configuration space [7].

2.2.1 In our situation

$$\mathcal{L}(x) = \tfrac{1}{2}\lambda^{-1}x^2 - V(x)$$

will be an analog of the Lagrangian function of physics. Its first term represents the quadratic term in the exponent of the background Gaussian integral, while

$$V(x) = \sum_{k \geq 0} b_k \frac{x^k}{k!}$$

plays the role of an 'interaction potential'; but we will allow coefficients

$$b_k = \sum_{g \geq 0} b_{g,k}\kappa^g \in \mathbb{R}[[\kappa]]$$

which are power series in the asymptotic parameter. However, we need to require that $b_{0,0} = b_{0,1} = b_{0,2} = b_{1,0} = 0$, ie that the coefficients $b_{g,k} = 0$ unless $2(g-1) + n > 0$ (or, alternately: $V(0) \equiv 0 \bmod \kappa^2$, $V'(0) \equiv V''(0) \equiv 0 \bmod \kappa$); this signals a secret connection with moduli spaces of Riemann surfaces [9, 13]. This extra generality will not be used in the main example in §3, but it illustrates some of the flexibility of this class of models.

2.2.2 Let $Z(0)$ denote $\int_{\mathbb{R}} \exp(-\kappa^{-1}\mathcal{L}(x))\ dx$ (or, more precisely, its representation as an asymptotic series); then we can define the 'expectation value' of a function f on the line to be

$$\langle f \rangle \ \sim\ Z(0)^{-1} \int_{\mathbb{R}} f(x)\ \exp(-\kappa^{-1}\mathcal{L}(x))\, dx\ .$$

If we now write

$$\beta_{g,k} = -\log b_{g,k}\ ,$$

then Wick's theorem takes the form of an asymptotic expansion

$$\langle \exp(\kappa^{-1}tx) \rangle \ \sim\ e^{\frac{1}{2}\kappa^{-1}\lambda t^2}[\ 1\ +\ \sum_{G \in |W\Gamma^+_\bullet|} \mathcal{P}_\beta(G)\kappa^{-\chi(G)}\lambda^{\varepsilon(G)} \frac{t^{\nu(G)}}{\nu(G)!}\ ]$$

(independent of the domain of integration, as long as it contains the origin). The sum is taken over the set $|W\Gamma^+_\bullet|$ of weighted 'non-vacuum' graphs: connected or not, but such that each component has at least one leg. From now on the leading term in right-hand expression will

be included in this sum, on the grounds that it can be regarded as the contribution of the empty set, understood as a non-vacuum graph.

Note that when $V = 0$, this formula reduces to the classical fact that the Fourier-Laplace transform of a Gaussian function is again Gaussian.

2.2.3 We can use some formal version

$$\langle f \rangle = \int_{\mathbb{R}} \hat{f}(\xi) \langle \exp(i\xi x) \rangle \, d\xi$$

of Plancherel's theorem to extend this result. Since the asymptotic expression depends only on the germ of f at the origin, we will regard $f \in \mathbb{R}[[x]]$ as a power series; substituting $t = i\kappa\xi$ gives

$$\langle f \rangle \;\sim\; 2\pi \sum_{|W\Gamma^+_\bullet|} \partial^\nu \exp(\tfrac{1}{2}\kappa\lambda\partial^2) \; f|_{x=0} \cdot \mathcal{P}_\beta \lambda^\varepsilon \kappa^{-\chi+\nu}/\nu! \; .$$

If $\gamma_k(x) = x^k/k!$ is the kth divided power of x, for example, then the coefficient

$$\partial^\nu \exp(\tfrac{1}{2}\kappa\lambda\partial^2) \; \gamma_k(x)|_{x=0} \;=\; \gamma_m(\tfrac{1}{2}\kappa\lambda)$$

if $k - \nu = 2m$ is even, and is zero otherwise. Note that the resulting "measures' are really formal sums of delta-functions and their derivatives, all with support at the origin!

2.2.4 As formulated in [10], Wick's theorem in one dimension asserts that

$$(2\pi\kappa\lambda)^{-1/2} \int_{\mathbb{R}} \exp \kappa^{-1} \Big(tx - \tfrac{1}{2}\lambda^{-1}x^2 + V(x) \Big) \, dx \;\sim\; \exp \kappa^{-1} \Big(\tfrac{1}{2}\lambda t^2 + \hat{V}(t) \Big) \; ,$$

where the formal transform

$$\hat{V}(t) = \sum \hat{b}_{\chi,\nu} \; \kappa^g \; \frac{t^\nu}{\nu!}$$

has coefficients

$$\hat{b}_{\chi,\nu} = \sum_{G \in |\Gamma(\chi,\nu)|} |\mathrm{Aut}(G)|^{-1} \lambda^{\varepsilon(G)} \prod_{v \in \mathrm{Ver}^0(G)} b_{w(v),k(v)} \; .$$

The term $\frac{1}{2}\lambda t^2$ will sometimes be interpreted as the contribution to $\hat{V}$ from the unique graph in $\Gamma(1, 2)$ (whose coefficient is not defined by the prescription above).

The expansion in §2.2.2 just rewrites this: when $t = 0$,

$$(2\pi\kappa\lambda)^{-1/2} \int \exp(-\kappa^{-1}\mathcal{L}(x)) \, dx \;\sim\; \exp(\kappa^{-1}\hat{V}(0)) \; ,$$

with $\hat{V}(0) = \sum_{g>1} \hat{b}_{g,0}\kappa^g$ (since $2(g-1)+n>0$). This is a sum over 'vacuum graphs' (without legs), so the 'renormalized' sum $\hat{V}_+(t) = \hat{V}(t) - \hat{V}(0)$ contains contributions only from graphs with at least one leg. Thus

$$\langle \exp(\kappa^{-1}tx)\rangle = Z(0)^{-1}\int \exp\kappa^{-1}(tx - \mathcal{L}(x))\,dx \;\sim\; \exp\kappa^{-1}(\tfrac{1}{2}\lambda t^2 + \hat{V}_+(t))\ ,$$

and the formula in §2.2 comes from expanding the exponential of $\kappa^{-1}\hat{V}_+$: we get

$$\prod_{|\Gamma(\chi,\nu)|,\nu\geq 1} \exp(|\mathrm{Aut}|^{-1} P_\beta \kappa^{-\chi}\lambda^\varepsilon \frac{t^\nu}{\nu!}) \;=\; \sum\prod \frac{[P_\beta(G_i)\kappa^{-\chi(G_i)}\lambda^{\varepsilon(G_i)}t^{\nu(G_i)}]^{k_i}}{|\mathrm{Aut}(G_i)|k_i!\;\nu(G_i)!^{k_i}}$$

which, everything being more or less additive, rearranges into

$$\sum \frac{\nu(G)!}{|\mathrm{Aut}(G)|\prod \nu(G_i)!^{k_i}} P_\beta(G)\kappa^{-\chi(G)}\lambda^{\varepsilon(G)}\frac{t^{\nu(G)}}{\nu(G)!} = \sum_{[W\Gamma^+_\bullet]} \mathcal{P}_\beta\kappa^{-\chi}\lambda^\varepsilon \frac{t^\nu}{\nu!}$$

(where $G = \sum k_i G_i$, with the G_i distinct).

2.3 We can summarize this elementary calculation as follows:

We consider two formal measure spaces,

- the real line $\mathbb{R}$, with respect to the formal measure

$$\mu_{\mathcal{L}} \sim (2\pi\kappa\lambda)^{-1/2}\int_{\mathbb{R}} \exp(-\kappa^{-1}\mathcal{L}(x))\,dx\ ,$$

(supported at 0, as noted above), and

- $|W\Gamma_\bullet|$, with respect to the measure $\mu_{\beta,\lambda}$ defined by summing $\mathcal{P}_\beta\lambda^\varepsilon\kappa^{-\chi+\nu}/\nu!$.

Let X_ν be the characteristic function of the set of non-vacuum weighted graphs with exactly ν legs, and define

$$\Phi_k = \sum_{m\geq 0} \gamma_m(\tfrac{1}{2}\kappa\lambda)X_{k-2m}\ .$$

With this notation, we have the

Theorem: The linear operator

$$f = \sum f_k\gamma_k \mapsto \check{f} = \sum f_k\Phi_k$$

maps the space $L^1(\mathbb{R},\mu_{\mathcal{L}})$ of formal functions isometrically to the formal span of the X_ν in $L^1(|W\Gamma_\bullet|,\mu_{\beta,\lambda})$.

This correspondence sees functions on the line only in the neighborhood of zero, and their transforms lie in the subspace spanned by the characteristic functions X_ν; but we can nevertheless probe the space of graphs in more detail, by varying the parameters λ and β.

3. Ensembles of scale-free trees

This general framework seems very rich, even after rather drastic specialization. This section is an account of some measures on spaces of trees defined by potentials

$$V(x) = \sum_{k \geq 3} \frac{x^k}{k^{\alpha+1}} \quad (= \mathrm{Li}_{\alpha+1}(x) - x - 2^{-\alpha-1}x^2)$$

of polylogarithmic form. [The 'missing' linear and quadratic terms are tracked by the parameters t and λ^{-1}; the conventions used here seem to simplify book-keeping in the long run [9].] The parameter κ does not appear in this formula, so we will be concerned with **un**weighted graphs; in fact we will be mostly concerned with trees (ie graphs of genus zero, which however will not be rooted).

This subject has a rather extensive literature in condensed-matter physics, closely related to the study of 'scale-free' networks [3,4,12], but one of the purposes of this note is to argue that it deserves more attention from mathematicians. In particular, it seems to me that this model is potentially at least as rich as (for example) the Ising model. It may also be useful to note that related path integral techniques have found other interesting applications in mathematical biology, eg [17].

3.1 The partition function for weighted graphs is the power-series expansion of an analytic function

$$Z(t,\lambda) = (2\pi\kappa\lambda)^{-1/2} \int_{\mathbb{R}} \exp(-\kappa^{-1}(tx - \mathcal{L}(x)))\, dx$$

near $t = \lambda = 0$. Following ideas pioneered by Landau [1, 14 Ch VI], we can try to interpret its singularities in terms of phase transitions.

The leading term in the statistical free energy for this ensemble of networks is the tree approximation

$$W(t) = \lim_{\kappa \to 0} \kappa \log Z \;\sim\; \tfrac{1}{2}\lambda t^2 + \lim_{\kappa \to 0} \hat{V}(t) = \sum_{G \in |\Gamma(1,\nu)|, \nu \geq 2} P_\beta(G) \lambda^\varepsilon \frac{t^\nu}{\nu!}\ ;$$

the last equality uses the fact that labeled trees have no automorphisms. We can omit the factorial in the denominator of the last term by writing it as a sum over isomorphism classes of **un**labeled trees.

3.2 A formal analog of Laplace's method for evaluating integrals with large parameter (ancestral to the saddle point or 'stationary phase' approximation), at a critical point x_0 of

$$\phi(x) = \phi(x_0) + \tfrac{1}{2}\phi''(x_0) \cdot (x - x_0)^2 + \dots \ ,$$

is an expression of the form

$$(2\pi\kappa)^{-1/2} \int_{\mathbb{R}} \exp(-\kappa^{-1}\phi(x))\, dx = \frac{\exp(-\kappa^{-1}\phi(x_0))}{\phi''(x_0)^{1/2}} \, [1\text{+higher order in } \kappa \dots] \ .$$

This implies a formula

$$W(t, \lambda) = t x_0 - \mathcal{L}(x_0)$$

for the tree level free energy, interpreted as a function of t, λ via the equation $t = \mathcal{L}'(x_0)$ for a critical point of $tx - \mathcal{L}(x)$. Rewriting that equation in iterable form yields

$$x_0(t, \lambda) = \lambda(t + V'(x_0)) = \lambda t + \lambda V'(\lambda t) + \dots \in \mathbb{R}[[t, \lambda]] \ ;$$

more generally, the implicit function theorem guarantees the existence of a solution $x = x_0(t, \lambda)$ off the critical locus $\lambda^{-1} = V''(x_0)$, ie away from

$$t_0 = x_0 V''(x_0) - V'(x_0) \ .$$

3.3 However, the polylogarithm has a subtle singularity at $x = 1$:

$$\mathrm{Li}_{\alpha+1}(e^{-x}) \ \equiv \ \Gamma(-\alpha)(x \sim i0)^{\alpha}$$

modulo smooth functions, where

$$(x \sim i0)^{\alpha} := \tfrac{1}{2}[(x + i0)^{\alpha} + (x - i0)^{\alpha}] = x_+^{\alpha} + \cos \pi\alpha \ x_-^{\alpha}$$

is the average of two possible Gel'fand-Shilov regularizations of the complex power. [The first draft of this paper contained a very confusing sign error here; I'm very sorry about that. A fuller discussion of this formula has been since posted at [6 Prop. 3].] Thus, if the potential has the form specified at the beginning of this section, then at $x = 1$ we recover equation (26)

$$t_0 = V''(1) - V'(1) = 1 + \zeta(\alpha - 1) - 2\zeta(\alpha)$$

of [3] for the locus of singularities in the model.

As α varies, this defines an approximate hyperbola in the positive quadrant of the (t, α) plane, partitioning it into two regions, as in Fig. 6

of [4]. Left of this line, the resulting measure is expected to be concentrated on trees with an exponential degree distribution, while to its right it is thought to be supported on 'clumpy' concentrations, with many edges abutting on a relatively small number of hubs. The critical line defines a phase transition, with the measure concentrated on the **scale-free** trees.

The behavior of x_0 as a function of λ near the critical line varies, depending on whether α is greater or less than two. If $\alpha > 3$, the first coefficient

$$\frac{\partial\lambda}{\partial x_0} = -x_0^{-2}(t + V'(x_0)) + x_0^{-1}V''(x_0)$$

in the Taylor series for $\lambda = \lambda(x_0)$ goes to zero near the critical line, so

$$x_0 = 1 - C(\lambda_0 - \lambda)^{1/2} + \cdots \ .$$

On the other hand the saddle-point equation

$$\lambda^{-1} = x_0^{-1}(t + V'(x_0))$$

can be rewritten

$$\lambda^{-1} = \lambda_0^{-1} + x_0^{-1}(t + V'(x_0)) - (t_0 + V'(1)) \ ,$$

(with $\lambda_0 = V'(1)$); as $t \to t_0$ this specializes to

$$\lambda^{-1} = \lambda_0^{-1} + (x_0^{-1} - 1)V'(1) + x_0^{-1}(V'(x_0) - V'(1)) \ .$$

When $3 > \alpha > 2$,

$$\lambda^{-1} \sim \lambda_0^{-1} + \Gamma(1-\alpha)(1 - x_0)^{\alpha-1} + \cdots \ ,$$

so

$$x_0 \sim 1 - C(\lambda_0 - \lambda)^{1/(\alpha-1)} + \cdots \ .$$

It is tempting to take

$$\eta = \log x_0 \sim (\lambda_0 - \lambda)^{1/(\alpha-1)}$$

(when $\alpha \in (2,3)$) as an order parameter for this transition [1,19 I §3].

Some further results about the higher-genus case can be found in [12]; I hope to return to these, and other questions about the generalized thermodynamical properties of these models, soon.

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